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On the Regularity of the Multifractal Spectrum of Bernoulli Convolutions

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In previous work we developed a thermodynamic formalism for the Bernoulli convolution associated with the golden mean, and we obtained by perturbative analysis the existence, regularity, and strict convexity of the pressure $F(\beta)$ in a neighborhood of $\beta=0$. This gives the existence of a multifractal spectrum $f(\alpha)$ in a neighborhood of the almost sure value $\alpha=f(\alpha)=0, 9957...$. In the present paper, by a direct study of the Ruelle–Perron–Frobenius operator associated with the random unbounded matrix product arising in our problem, we can prove the regularity of the pressure $F(\beta)$ for (at least) $\beta\in(-1/2, +\infty)$. This yields the interval of the singularity spectrum between the minimal value of the dimension of ν, $\alpha_{\min}=0.94042...$, and the almost sure value, $\alpha_{a.s.}=0.9957...$.

KEY WORDS: Random matrices; thermodynamic formalism; multifractal analysis.

INTRODUCTION

Let $\varepsilon_1, \varepsilon_2,...$ be a sequence of independent random variables each taking the values $+1$ and -1 with equal probability. The probability distribution of the random variable $(1-\gamma)\sum_{n=0}^{\infty}\varepsilon_n\gamma^n$, $0<\gamma<1$, defines a measure ν_γ which is called an Infinitely Convolved Bernoulli Measure or simply a Bernoulli Convolution. For $\gamma>1/2$ it is a difficult, old and not yet completely solved problem to decide on the nature of ν_γ. P. Erdős proved the singular continuous nature of ν_γ if γ^{-1} is a Pisot number (that is an algebraic integer whose conjugates lie inside the unit circle). Recently

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B. Solomiak proved that for almost all $\gamma \in [1/2, 1]$ ν_γ is absolutely continuous (see ref. 5 for more details and references).

In ref. 5 we concentrated on the singularity of ν_γ in the case where γ the golden mean. We first studied, following Young,[9] the existence of the ν_γ almost sure limit

$$\lim_{\substack{|I| \to 0 \\ x \in I}} \frac{\log \nu_\gamma(I(x))}{\log |I(x)|} = \delta \tag{1}$$

where $I(x)$ is an interval centered at x, and our first result was an explicit formula for the Hausdorff dimension of the measure ν_γ, namely $HD(\nu_\gamma) = \delta$.

Following Alexander and Yorke, we introduced the (non-inversible) map $(x, y) \to T(x, y)$:

$$T(x, y) = \begin{cases} \dfrac{x}{\gamma}, \quad 2y & \text{if} \quad y \leqslant 1/2, \quad x \leqslant \gamma \\ \dfrac{x}{\gamma} - \gamma, \quad 2y - 1 & \text{if} \quad y \geqslant 1/2, \quad x \geqslant 1 - \beta \end{cases} \tag{2}$$

with $\gamma + \gamma^2 = 1$.

Note that this model offers a situation which is very different from the Axiom A case for example: T is an endomorphism of the square which possesses two expanding directions.

Let $A = [1-\gamma, \gamma] \times [1/2, 1]$, $B = [\gamma, 1] \times [1/2, 1]$, $C = [0, 1-\gamma] \times [0, 1/2]$, $D = [1-\gamma, \gamma] \times [0, 1/2]$. Since $\gamma + \gamma^2 = 1$, $P_0 \equiv \{A, B, C, D\}$ is a Markov partition with compatibilty rules: $A \to C$; $B \to A, B, D$; $C \to A, C, D$; $D \to B$. That is, every point $(x, y) \in X$ is coded by a sequence $\underline{a}(x, y) = a_0 a_1 \cdots$ with $a_i \in \{A, B, C, D\}$ such that $(x, y) \in a_0$, $T(x, y) \in a_1$,..., $T^n(x, y) \in a_n$,... and viceversa any compatible sequence $a_0 a_1 \cdots$ defines a unique point $(x, y) \in X$. The invariance relations: $\mu(T_0^{-1} I) = \frac{1}{2}\mu(I)$, $\mu(T_1^{-1} I) = \frac{1}{2}\mu(I)$ for $I \in [0, 1] \times [0, 1]$ and the above Markov compatibility rules uniquely define the maximum entropy Markov invariant measure μ.

One can prove that ν_γ is the transverse measure of the measure μ. We then studied the relations between the Markov partition P_0 for T and the γ-adic partition of the x-axis (i.e., the partition generated by the expansion of $x \in [0, 1]$ on the basis provided by the powers of γ). The ν_γ mesure of a γ-adic interval is computed by counting the rectangles of the Markov partition which project on it. The dimension of the measure ν_γ is therefore associated to the growth of a random products of Markov matrices. These (unbounded) matrices are

$$M(n) = \begin{cases} \begin{pmatrix} 1 & k+1 \\ 1 & k+1 \end{pmatrix} & \text{if} \quad n = 2k+2 \\ \begin{pmatrix} 1 & k \\ 1 & k+1 \end{pmatrix} & \text{if} \quad n = 2k+1 \end{cases}$$

and $M(n) \equiv M(x_n)$ where $x_n \in F^{-n} P_0$.

We then considered the space X^N of the trajectories $\{x_n\}_{n \in N}$ of the Markov process with distribution P_μ such that if $x_n(\underline{x}) = x_n$ then $P_\mu[x_n = i] = \mu(i)$ and $P_\mu[x_{n+1} = j \mid x_n = i] = \pi(j \mid i)$, where the initial distribution μ and the transition matrix $\pi(i \mid j)_{i \in N, j \in N}$ are described in ref. 5, and are such that if $n_1 \cdots n_q$ is the coding of a γ-adic interval, then its ν_γ-measure equals $|M(x_{n_q}) \cdots M(x_{n_1})| / 2^{(x_{n_1} + \cdots + x_{n_q})}$ and its length equals $l(x_{n_1}) + \cdots + l(x_{n_q})$ where $l(x_i) = \log \gamma^{|x_i| + 1}$. Our dimension formula, that is, the limit in (1) then appears in a natural way as a version of the Furstenberg–Guivarch formula, (cf. ref. 5, for more details).

Then we studied the local exponent:

$$\alpha(x) = \lim_{\substack{|I| \to 0 \\ x \in I}} \frac{\log \nu_\gamma(I(x))}{\log |I(x)|} \tag{3}$$

if the limit exists. Let $B_\alpha = \{x: \alpha(x) = \alpha\}$ and $f(\alpha)$ the Hausdorff dimension of B_α. Multifractal analysis is concerned with the study of $\{(\alpha, f((\alpha))\}$, the "dimension spectrum" of the measure ν_γ.

Consider a dynamical system (Ω, T, μ). Thermodynamic formalism[8] provides a by now "classical" method[3] to analyze its dimension spectrum. Let $Z_n = \sum_{I \in A_n} \nu_\gamma(I)^\beta$, where A_n is an exponentially fast (with n) decreasing partition of the system, and let us assume that the thermodynamic limit $\lim_{n \to \infty} (1/n) \log Z_n$ exist and define a regular function ("pressure") $F(\beta)$. Then, if we denote by $\tilde{f}(\alpha)$ the Legendre transform of $F(\beta)$, that is $\inf_\beta(\alpha\beta - F(\beta))$, then the large deviations theorem states that $\#\{I: \nu_\gamma(I) \sim |I|^\alpha\}$ behaves as $\exp(n\tilde{f}(\alpha))$, for large n. This result would allow us to show that actually $\tilde{f}(\alpha) = f(\alpha)$, that is, $\tilde{f}(\alpha)$ is the Hausdorff dimension of B_α, the set where the measure has a power law singularity of strength α. This gives the meaning of $\tilde{f}(\alpha)$ in terms of μ, and moreover provides a method to compute $f(\alpha)$ as Legendre transform of $F(\beta)$. Our model does not allow us to work out estimates on the measure of uniform atoms and therefore we chose to consider a joint partition function $G_n(\beta, F) = \sum_{I \in A_n} \mu(I)^\beta \ell(I)^F$, the thermodynamic limit of which can also be studied via large deviations theorem. We dealed with a two-dimensional version of it, because of the joint fluctuations of masses and volumes. Consequently, the dimension of the set of trajectories where the measure has a singularity of strength α

turns out to be the Legendre transform $\tilde{f}(\alpha)$ of the (unique) function F realizing the "good" (mass/volume) section of the two-dimensional problem.

Note that $\tilde{f}(\alpha)$, while obtained as a section of a joint large deviation function $f(\alpha, \ell)$ is intrinsic to the dynamical system (Ω, f, μ). Indeed, if the pointwise limit

$$\lim_{\substack{|I| \to 0 \\ x \in I}} \frac{\log \mu(I(x))}{\log |I(x)|} = \alpha$$

exists and is equal to α on a set B_α of points x, then the limit exists and is the same for all sub-sequence of intervals containing $x \in B_\alpha$, whose diameter goes to zero. We can then associate to B_α its Hausdorff dimension $f(\alpha)$. Our main result on the multifractal analysis of the Bernoulli convolution was the following theorem (see ref. 5):

Theorem. Let

$$S(\alpha+\delta) = \left\{ \underline{x} \text{ such that } \lim_{n \to \infty} \frac{\log \dfrac{|M(x_n) \cdots M(x_0)\, u|}{2^{(x_0 + \cdots + x_n)}}}{\log(l(x_0) + \cdots + l(x_n))} = \alpha + \delta \right\}$$

and $pS(\alpha+\delta)$ its projection on $[0, 1]$, that is the set where the local exponent (3) of the measure ν_γ is $\alpha+\delta$. Then, for $|\alpha|$ sufficiently small

$$HD(pS(\alpha+\delta)) = \tilde{f}(\alpha+\delta) + (\alpha+\delta)$$

where $\tilde{f}$ is the Legendre transform of the above defined F.

There are very few examples where the mathematics of the multifractal spectrum is well understood: our model is perhaps the first for which it has been possible to obtain a result on multifractal analysis of ν_γ—and then of μ—for a repelling dynamical system in dimension two. However, the proof relies on perturbative analysis which is conclusive only near zero. As in classical random matrix products theory, we obtained the existence, regularity, and strict convexity of the pressure $F(\beta)$ only in a neighborhood of $\beta = 0$, that is, the existence of a multifractal spectrum $f(\alpha)$ in a neighborhood of the almost sure value $\alpha = f(\alpha) = 0{,}9957...$.

In the present paper, by a direct study of the Ruelle–Perron–Frobenius operator associated to the random unbounded matrix product arising in our problem, we can prove the regularity of the pressure $F(\beta)$ for (at least) $\beta \in (-1/2, +\infty)$. This yields, via the Legendre transformation, the interval of the singularity spectrum between the minimal value of the dimension of ν, $\alpha_{\min} = 0.94042...$ and the almost sure value $\alpha_{\text{a.s.}} = 0.9957...$.

This paper is organized as follows: in Section 1 we prove a Ruelle–Perron–Frobenius theorem, in Section 2 we derive the regularity and strict convexity of the pressure, and in Section 3 we apply these results to the multifractal analysis of the measure ν_γ.

1. A RUELLE-PERRON-FROBENIUS THEOREM

We keep the same notations as in ref. 5, and, as in ref. 5, we first set the theory in a simpler context, in order to simplify exposition, and then we extend it.

Let $L_{\alpha\gamma}$ the space of functions f on $X \times S$ (S is the circle) such that $\|f\|_{\alpha\gamma} < \infty$ where $\| \;\|_{\alpha\gamma} = \sup_{x_0, u} |f(x_0, u)|/|x_0|^\gamma + \sup_{x_0, u \neq v} |f(x_0, u) - f(x_0, v)|/|x_0|^\gamma \delta(u, v)^\alpha$. Let $T_\beta f(x_0, u) = E_{x_0} e^{\beta \log(|M(x_0) u|/2^{|x_0|})} f(x_1, M(x_0) u)$ with $\beta \in (-1, +\infty)$, $0 < \alpha < 1$ and $\gamma > 0$. We first study T_β for positive β. We are going to prove a Ruelle–Perron–Frobenius theorem for the operator T_β:

Theorem 1. Let $\beta \geqslant 0$. Let $T_\beta f(x_0, u) = E_{x_0} e^{\beta \log(|M(x_0) u|/2^{|x_0|})} \times f(x_1, M(x_0) u)$. Then:

(1) There exists a simple maximal eigenvalue $\lambda(\beta)$ of $T(\beta)$ with strictly positive eigenfunction h_β.

(2) The remainder of the spectrum of T is contained into a disk of radius strictly smaller than $\lambda(\beta)$.

(3) It existe an unique probability ν_β such that $T_\beta^* \nu_\beta = \lambda(\beta)\, \nu_\beta$.

(4) $\lambda^{-n}(\beta)\, T^n(\beta)\, f(x_0, u) \to h_\beta(x_0, u) \int_{X \times S} f(x_0, u)\, d\nu_\beta(x_0, u)$ in $L_{\alpha, \gamma}$.

Proof. The proof results of several lemmas.

Lemma 1.1. $\forall \beta \geqslant 0$ we have $\|T_\beta f\|_{\alpha\gamma} \leqslant \|f\|_{\alpha\gamma}$

Indeed let $|f(x_0, u) - f(x_0, v)| \leqslant |x_0|^\gamma \delta(u, v)^\alpha$. We look for a condition on β such that $|T_\beta f(x_0, u) - T_\beta f(x_0, v)| \leqslant |x_0|^\gamma \delta(u, v)^\alpha$. We have (cf. ref. 5):

$$|T_\beta f(x_0, u) - T_\beta f(x_0, v)| \leqslant E_{x_0}\left(\beta \frac{|M(x_0)\, u|^\beta}{2^{\beta x_0}} \delta(u, v)^\alpha\, |f(x_1, M(x_0)\, u)|\right)$$

$$+ E_{x_0}\left(\frac{|M(x_0)\, u|^\beta}{2^{\beta x_0}} |x_1|^\gamma\, \delta(M(x_0)\, u, M(x_0)\, v)^\alpha\right)$$

and we state $\leqslant \delta(u, v)^\alpha\, |x_0|^\gamma$

which gives the condition $|M(x_0)\,u|^{\beta}/2^{\beta x_0}\sum_{x_1}\pi(x_1\mid x_0)(\beta\;|f|_\infty+1)|x_1|^{\gamma}\leqslant |x_0|^{\gamma}$ or—if $\gamma\leqslant 1$ and $|f|_\infty\leqslant 1$—(cf. ref. 5) $(2x_0+4)^{\beta}(1+\beta)/4\leqslant 2^{4\beta}$ with $x_0\geqslant 4$, which is true for any $\beta>0$.

T_β being positive, we consider the operator on $M(X\times S)$: $\tilde{T}^*_\beta\mu=T^*\mu/T^*\mu(1)$.

Lemma 1.2. Let $K_a=\{(x_0,u)$ with $x_0\leqslant a,\ u\in S\}$. Let Γ the set of measures on $X\times S$ such that $\mu(K_a)\leqslant 1/a$ (with large a). Then T^*_β preserves Γ.

Indeed

$$T^*\mu(K_a)=\int_{X\times S}T_\beta 1_{K_a}(x_0,u)\,d\mu(x_0,u)$$

$$=\int_{X\times S}\sum_{x_1}\pi(x_1\mid x_0)\,e^{\beta\log(|M(x_0)\,u|/2^{|x_0|})}1_{K_a}(x_1,M(x_0)\,u)\,d\mu(x_0,u)$$

$$=\int_{X\times S}\sum_{k\geqslant a}\frac{c(x_0)}{2^k}\,e^{\beta\log(|M(x_0)\,u|/2^{|x_0|})}1_{K_a}(x_1,M(x_0)\,u)\,d\mu(x_0,u)$$

and as $|M(x_0)\,u|/2^{|x_0|}<1$ and $c(x_0)\leqslant 4$ this expression is less than $(1/2^{a-1})\times 4[\int_{K_a}d\mu+\int_{K_a^c}d\mu]\leqslant(C/2^{a-1})$. We so ask $C/2^{a-1}\leqslant 1/a$ id est $Ma<2^a$, which is $\forall a>a'\equiv\gamma\log M$ where γ satisfies $M^{\gamma}=M\gamma\log M$ (here, if $M=16$, this is true for $a>7$).

Remark. This lemma remains true also for $\beta\geqslant -1$, with minor modifications.

Γ is a convex compact subset of $M(X\times S)$, $\tilde{T}^*$: $\bar{\Gamma}\to\bar{\Gamma}$ is continuous. By the Schauder–Tychonoff theorem there exists $\nu\in\bar{\Gamma}$ such that $\tilde{T}^*\nu=\nu$ where $(T^*\nu/T^*\nu(1))=\nu$. Let $\lambda(\beta)=T^*_\beta\nu(1)$. We have $\lambda(\beta)>0$.

Lemma 1.3. Let $\Lambda=\{f\in C(X\times S)\ \|f\|_{\alpha\gamma}<\infty,\ \int f\,d\nu=1,\ f\geqslant 0\}$ there exist $h\in\Lambda$ such that $T_\beta h=\lambda(\beta)\,h$.

Indeed by Lemma 1 T_β is a contraction on Λ. Let $\Lambda_{x_0}=\{f\colon X\times S\to C$, $f(x_0,\cdot)$ is continuous on S, $\sup_u(|f(x_0,u)|/|x_0|^{\gamma})+\sup_{u\neq v}(|f(x_0,u)-f(x_0,v)|/|x_0|^{\gamma}\,\delta(u,v)^{\alpha})<\infty\}$. Λ_{x_0} is uniformly continuous and uniformly bounded. $\Lambda=\Pi_{x_0\in X}\Lambda_{x_0}$ is compact as countable product of the compact sets (by Ascoli–Arzelà) Λ_{x_0}. Let $\tilde{T}_\beta=(T_\beta/\lambda)\,\tilde{T}_\beta$: $\Lambda\to\Lambda$. Then there exists in Λ such that $\tilde{T}_\beta h=h$ (by Schauder–Tychonoff).

Lemma 1.4. $h(x_0,u)$ is strictly positive.

If there were a $(\bar{x}_0, \bar{u})$ such that $h(\bar{x}_0, \bar{u}) = 0$ then we had—as we are going to see—$h \equiv 0$ (and if $h \equiv 0$ from $0 < \lambda = (\langle Th, v\rangle / \langle h, v\rangle) = \int Th\, dv$ $= 0$ we had a contradiction). So, let $(\bar{x}_0, \bar{u})$ be such that $h(\bar{x}_0, \bar{u}) = 0$. Then $0 = Th(\bar{x}_0, \bar{u}) = \sum_{x_1} \pi(x_1 \mid x_0)\, e^{\beta \log(|M(\bar{x}_0)\,\bar{u}|/2^{|x_0|})} \times h(x_1, M(\bar{x}_0)\,\bar{u})$ which implies $h(x_1, M(\bar{x}_0)\,\bar{u}) = 0$ $\forall (x_1, M(\bar{x}_0)\,\bar{u} \equiv v)$ such that the transition $\bar{x}_0 \to x_1$, $\bar{u} \to v$ is allowed. Repeating this argument gives $h(x_0, u) = 0$ on a dense subset of $X \times S$, since our system has the following property (cf. ref. 5).

Property *: "$\forall \varepsilon > 0$, there exists $N \in N$ et $p > 0$ such that $\forall (x, u)$, $(\bar{x}_0, \bar{u})$ the N-step transition probability from $(\bar{x}_0, \bar{u})$ to (y, w) is larger than p and (y, w) is a point ε-near to (x, u)."

In fact, the system has a stronger property, which we shall need below, in constructing the invariant measure:

Property **: "$\forall \varepsilon > 0$, there exists $N \in N$ and $p > 0$ such that $\forall (x, u)$, $(\bar{x}_0, \bar{u})$ in the support of v_β the N-step transition probability from $(\bar{x}_0, \bar{u})$ to (x, u) is larger than p."

Lemma 1.5. h is simple.

If f and g are two eigenfunctions, then, $h = g - f$ has a positive and a negative part (if $g(x_0, u) > f(x_0, u)$ $\forall (x_0, u)$ then $0 = \int (g - f)\, dv$ (since $\langle f, v\rangle = \langle g, v\rangle = 1$) with $g \geqslant f$ give $g \equiv f$). Let $A = \{h > 0\}$ and $B = \{h < 0\}$. By using the above property (* or **) we see that A and B are in fact two invariant, disjoint full measure sets, contradicting the ergodicity. (indeed, ergodicity comes from the fact that $v_\beta \equiv \pi(x_0)\, v_{x_0, \beta}(u)$ is a measure on $X \times S$, such that $\pi(x_0)$ is the only stationary measure for Π_{ij} (Doeblin, see ref. 5) and $v_{x_0, \beta}(u)$ is discrete, with a dense orbite in S).

Let, by now, $\tilde{T}_\beta f = (\lambda^{-1} T_\beta h f / h)$. Clearly, $\tilde{T}_\beta 1 = 1$.

Lemma 1.6. (inequality à la Doeblin–Fortet)

$$m_{\alpha\gamma}(T_\beta^n f) \leqslant c(\beta)\, |f|_\infty + \rho^n m_{\alpha\gamma}(f)$$

where $\rho < 1$ and $m_{\alpha\gamma}(f) = \sup_{x,\, u \neq v}(|f(x, u) - f(x, v)| / |x|^\gamma\, \delta(u, v)^\alpha)$ and $|f|_\infty = \sup_{x, u}(|f(x, u)| / |x|^\gamma)$.

This estimation is proven in ref. 5.

We have to prove now that:

2′ The remainder of the spectrum is contained in a disk of radius strictly < 1.

3′ There exists only one v: $\tilde{T}^* v = v$.

4′ $\tilde{T}_\beta^n f \to \int f\, dv$ in A.

We start by proving the convergence 4′. Note first that:

$4'_1$ $\sup f(x_0, u) \geqslant \sup \tilde{T}f(x_0, u)$ (since $\tilde{T}$ is a contraction in Λ).

Let $f^* = \lim_{k\to\infty} T_\beta^{n_k} f$, then $f^* \in \Lambda$ if $f \in \Lambda$, and we have:

$4'_2$ $\sup T_\beta^n f^* = \sup f^*$

To end, we prove that:

$4'_3$ $f^* = \text{constant}$.

We distinguish two cases:

(1) If the supremum is attained on $X \times S$, $\sup f^* = f^*(\bar{x}_0, \bar{u}) = \sup T^N f^* = T^N f^*(\bar{x}_N, \bar{u}_N)$, In this case, $\sup T^N f^* = T^N f^*(\bar{x}_N, \bar{u}_N) = \sum e^{S_N(y,v)} f^*(y, v) = f^*(\bar{x}_0, \bar{u}) = \sup f^*$, where the sum runs over the (y, v) such that the N-step transition probability from $(\bar{x}_N, \bar{u}_N)$ to (y, v) is positive, and $S_N(y, v) \equiv M(y) \cdots M(\bar{x}_N)\, v$. But since $\sum e^{S_N(y,v)} 1 = 1$, and $0 \leqslant f^* \leqslant \sup f^*$, this implies $f^*(y, v) = \sup f^*$ $\forall (y, v)$ such that the N-step transition from (x_N, u_N) to (y, v) has a positive probability. But this set is dense in $X \times S$ (property *) so that $f^* = \text{constant}$, (since $f \in \Lambda$).

(2) If the supremum is not attained on $X \times S$, one can show that:[10]

Claim. $\forall \varepsilon > 0$, if $\delta > 0$ then $f^*(x, u) > \sup f^* - \delta$ for $(x, u) \in$ a set ε-dense on $X \times S$. (so, $f^* = \text{constant}$).

We use the property *. Let $(\bar{x}_N, \bar{u}_N) \in \bar{X} \times S$ the point of max for $T^N f^*$, and let $(z_N, v_N) \in X \times S$ sufficiently near the point of maximum. Then there exist N and $p > 0$ such that the N-step transition from (x_N, u_N) to (y, v) has a probability larger than p and (y, v) is sufficiently near to an arbitrarily chosen point (x, w). (N and p being uniform with respect to this choice). This follows from the property *. Also, it follows that, $e^{\beta \log M(y) \cdots M(x_N)\, \bar{u}_N} \pi^N(\bar{x}_N \to y) > b$ with $b > 0$, where $M(y) \cdots M(x_N)\, \bar{u}_N \equiv v$. The set of (x, u) such that the N-step transition from (z_N, v_N) to (x, u) has probability > 0 is dense. Let us show that for such (x, u) we have $f^*(x, u) > \sup f^* - \delta$. If on the contrary there were a (y, v) (in this dense set) such that

(1) $f^*(y, v) \leqslant \sup f^* - \delta$

(2) $e^{\beta \log M(y) \cdots M(z_N)\, v_N} \pi^N(z_N \to y) > b$ with $M(y) \cdots M(z_N)\, v_N \equiv v$.

which we will rewrite in short by:

(1) $f^*(\underline{y}) \leqslant \sup f^* - \delta$

(2) $e^{\beta \log S_n(\underline{y})} \pi^N(\underline{y}) > b$

and if we also denote by $E_{z_N}(e^{\beta \log S_n(u)}\pi^N(u)\, f^*(u)\, h(u))/(\lambda_\beta h(z_N))$ the expression for $\tilde{T}^N_\beta f(z_N, v_N)$ we should have:

$$\tilde{T}^N_\beta f(z_N, v_N) \equiv E_{z_N} \frac{e^{\beta \log S_n(y)}\pi^N(u)\, f^*(u)\, h(u)}{\lambda_\beta h(z_N)}$$

$$\leqslant \max f^*(1 - e^{\beta \log S_n(y)}\pi^N(y)\, f^*(y)\, h(y))$$

$$\times (\max f^* - \delta)(e^{\beta \log S_n(u)}\pi^N(y)\, h(y)) \qquad \text{since } (1)$$

$$\leqslant \max f^* - \delta b H \qquad \text{since } (2)$$

where $H = \inf h > 0$ and having used the inequality $\sum_i \lambda_i(x_i - x_j\lambda_j) \leqslant \max_i x_i(1-\lambda_j)$ where $\sum_i \lambda_i = 1$, $\lambda_i \geqslant 0$. Which is absurd being $\tilde{T}^N_\beta f(z_N, v_N)$ as near as we wish to $\sup \tilde{T}^N_\beta f(z_N, v_N) = \sup f^*$

Corollary 1.7. $f^* = \int f\, dv$. Moreover, v is unique

We end with the Proof of 2′ (the remainder of the spectrum): 2′ is true iff the spectral radius of $\tilde{T}_\beta$ restricted to the functions $f \in L_{\alpha\gamma}$ with $\int f\, dv = 0$ is strictly smaller than 1. But this follows by Lemma 6, which gives, for N and k large and f with zero average, the estimate $\|\tilde{T}^{N+k}_\beta f\|_{\alpha\gamma} \leqslant \varepsilon$. Indeed $m_{\alpha\gamma}(T^{N+k}_\beta f) \leqslant c(\beta)\, |T^k f|_\infty + \rho^n m_{\alpha\gamma}(T^k f) \leqslant c(\beta)\, |T^k f|_\infty + \rho^N[c(\beta)\, |f|_\infty + \rho^k m_{\alpha\gamma}(f)]$. But, since $|T^k f - \int f\, dv|_\infty$ goes to 0 when $k \to \infty$, if f has zero average we will have $|T^k f|_\infty \to 0$ so that if $\rho < 1$, for k and N large, $\|\tilde{T}^{N+k}_\beta f\|_{\alpha\gamma} = m_{\alpha\gamma}(T^{N+k}_\beta f) + |T^{N+k} f|_\infty \leqslant \varepsilon$.

2. STRICT CONVEXITY OF THE PRESSURE

In this paragraph we show that $P(\beta) = \lim(1/n) \log T^n_\beta 1$ is C^2 and strictly convex (Proposition 2.9 below)

Lemma 2.1. $\beta \to \tilde{T}_\beta$ is of class C^2 on $L_{\alpha\gamma}$.

Indeed, as in ref. 5 one has that $\|(d/d\beta^2)\, \tilde{T}_\beta f\|_{\alpha\gamma} \leqslant c$ if $\beta > 0$.

Lemma 2.2. (Spectral theorem) Let $\tilde{T}_{\beta_0}$ a class C^2 operator on $L_{\alpha\gamma}$ for $\beta_0 > 0$. let $\lambda(\beta_0)(=1)$ isolated simple eigenvalue of $\tilde{T}$ with eigenfunction 1. Then one has the spectral projector N on the eigenspace of 1: $T(\beta_0) = \lambda(\beta_0)\, N(\beta_0) + Q(\beta_0)$, $QN = NQ = 0$.

Then $\forall \beta$ positive we consider $s \to T(\beta_0 + s\beta)$. There exists a projector $N(s)$ such that $N(s)\, 1 = h(s)$ and $h(s)$ is eigenfunction of $T(\beta_0 + s\beta)$ with eigenvalue $\lambda(\beta_0 + s\beta)$:

$$T(\beta_0 + s\beta) = \lambda(\beta_0 + s\beta)\, N(s) + Q(s) \qquad N(s)\, Q(s) = Q(s)\, N(s) = 0$$

$T(\beta_0+s\beta)\,N(s)\,1=\lambda(\beta_0+s\beta)\,N(s)\,1$, $h(s)$ and $\lambda(s)$ are of class C^2 $\forall\beta>0$.

Lemma 2.3. (beauty of ν) Let $P(\beta_0+s\beta)\equiv\log\lambda(\beta_0+s\beta)$. Then

$$P'(0)=\beta\int\log\frac{M(x_0)\,u}{2^{x_0}}\,d\nu_\beta(x_0,u)$$

Indeed, it is sufficient to differentiate the relation $T(\beta_0+s\beta)\,N(s)\,1=\lambda(\beta_0+s\beta)\,N(s)\,1$ at $s=0$.

Lemma 2.4. (Ergodic theorem) $P_\pi(\underline{x})\times\nu_\beta(x_0,u)$ almost surely

$$\frac{1}{n}\log\frac{M(x_n)\cdots M(x_0)\,u}{2^{x_n+\cdots x_0}}\to\beta\int\log\frac{M(x_0)\,u}{2^{x_0}}\,d\nu_\beta(x_0,u)$$

(on the dynamical system $(X^N\times S,\ ((x)_n\to(x)_{n+1},\ S_n(x,u)\to S_{n+1}(x,u)),\ P_\pi(\underline{x})\times\nu_\beta(x_0,u)))$. where $S_n(x,u)=M(x_n)\cdots M(x_0)\,u$.

Differentiating two times the relation $T''(\beta_0+s\beta)\,N(s)\,1=\lambda''(\beta_0+s\beta)\times N(s)\,1$ at $s=0$, gives the

Lemma 2.5.

$$P''(0)=\lim_{n\to\infty}\frac{1}{n}\,\beta^2\int E_{x_0}\left(\log\frac{S_n(x,u)}{2^{x_0+\cdots x_n}}\right)^2 d\nu_\beta(x_0,u)$$
$$+\frac{2}{n}\,\beta\int E_{x_0}\log\frac{S_n(x,u)}{2^{x_0+\cdots x_n}}\,h'(0)(x_0,u)\,d\nu_\beta(x_0,u)$$

Lemma 2.6. (Ergodic theorem)

$$\lim_{n\to\infty}\frac{1}{n}\int E_{x_0}\left(\log\frac{S_n(x,u)}{2^{x_0+\cdots x_n}}\right)h'(0)\,d\nu_\beta(x_0,u)$$
$$=\int\log\frac{M(x_0)\,u}{2^{x_0}}\,d\nu_\beta(x_0,u)\int h'(0)(x_0,u)\,d\nu_\beta(x_0,u)$$
$$=P'(0)\int h'(0)(x_0,u)\,d\nu_\beta(x_0,u)$$

Let $\sigma^2=\lim(1/n)\int E_{x_0}(\log(S_n(x,u)/2^{x_0+\cdots x_n})-nP'(0))^2\,d\nu_\beta(x_0,u)$. As in ref. 5 we have

Proposition 2.7. $P(\beta)$ is C^2 and strictly convex $\forall\beta > 0$.

Indeed one has as in ref. 5 $\sigma^2 = (\partial^2 P/\partial\beta^2) > 0$. The proof is exactly the same but one plays with ν_β (instead of ν). Indeed we remark that que $\nu_\beta(x_0, u)$ is again esplicit and we find by differentiating the invariance equation $T^*\nu = \nu$ (cfr ref. 5) that $\nu_\beta(x_0, u) = \pi(x_0)\, \tilde{\nu}_{x_0}(u, \beta)$, where the equation for $\tilde{\nu}$ is

$$\tilde{\nu}_{x_1}(\phi, \beta) = \lambda(\beta)^{-1} \frac{2}{\mu(x_1)} \sum_{x_0} \mu(x_0)\, \pi(x_0, x_1)\, e^{\beta \log(|M(x_0)\,\phi|/2^{|x_0|})} \tilde{\nu}_{x_0}(M^{-1}(x_0)\,\phi)$$

where $x_0, x_1 \in X$ and $u, \phi \in S$.

Remark. (β negative) If $0 > \beta > -1$ we consider the norm $\|\,|\;|\,\|_{\alpha\gamma} = \sup_{x_0, u}(|f(x_0, u)|/2^{x_0}) + \sup_{x_0, u \neq v}(|f(x_0, u) - f(x_0, v)|/|x_0|^\gamma\, \delta(u, v)^\alpha\, 2^{x_0})$, and we can prove that T is a contraction for this norm if $|\beta| < \gamma < 1/2$. Indeed with this choice we satisfy to the condition (analogue to Lemma 1: $|T_\beta f(x_0, u) - T_\beta f(x_0, v)| \leqslant 2^{\gamma x_0}\, \delta(u, v)^\alpha$. Similarly we obtain that $P(\beta)$ is regular and strictly convex for $|\beta| < 1/2$. Now, P is independent on the chosen norm, as it is the $\lim_{N\to\infty} T^N 1$. In conclusion:

Proposition 2.8. $F(\beta)$ is C^2 and strictly convex (at least) for $\beta \in (-1/2, \infty)$.

3. MULTIFRACTAL ANALYSIS

In this paragraph we apply the thermodynamic formalism to the multifractal analysis of Bernoulli convolutions. As in ref. 5, the operator T_β is not sufficient, and we have to introduce a "joint" operator

$$T(\beta_1, \beta_2)\, f(x_0, u) = E_{x_0} e^{\beta_1 \log |M(x_0)\, u|/|u|} e^{-\beta_1 g(x_0)} e^{\beta_2 l(x_0)} f(x_1, M(x_0)\, u$$

where $g(x_0) = \log 2^{\|x_0\| + 1}$ and $l(x_0) = \log \gamma^{\|x_0\| + 1}$

Definition 3.1.[5] Let θ_1, θ_2 two positive real numbers. Let $L_{\eta, \theta_1, \theta_2}$ be the space of functions $f: X \times S \to C$ such that $\|f\|_{\eta, \theta_1, \theta_2} < \infty$ where $\|f\|_{\eta, \theta_1, \theta_2} = \sup_{x_0, u}(|f(x_0, u)|/2^{\theta_1 \|x_0\|}(\|x_0\| + 1)^{\theta_2}) + \sup_{x_0, u \neq v}(|f(x_0, u) - f(x_0, v)|/2^{\theta_1 \|x_0\|}(\|x_0\| + 1)^{\theta_2}\, \delta(u, v)^\eta)$.

The preceeding theory applies and we similarly obtain the following proposition: (cf. ref. 5), by using the spectral theorem and the implicit function theorem:

Proposition 3.2. $T(\beta_1, \beta_2)$ is a family of class C^2 of bounded operators of $L_{\eta, \theta_1, \theta_2}$ for $\beta_1 \in (-1/2, +\infty)$ and $\beta_2 > 0$. Moreover, let $G(\beta_1, \beta_2) = \lim_{n \to \infty} (1/n) \log T^n(\beta_1, \beta_2)\, 1(x_0, u)$. Then there exists a function F, defined for $\beta \in (-1/2, +\infty)$, of class C^2, such that $F(0) = 0$, $G(\beta, F(\beta)) = 0$.

Definition 3.3. Let $f(\alpha + \delta) = \sup_{\beta \in (-1/2, \infty)} \{(\alpha + \delta)\beta - F(\beta)\}$ where $\delta = (\lambda - E \log 2/E \log \beta) = HD(v_\gamma)$—where v_γ is the Bernoulli convolution of the golden mean. Then via the Legendre conjugation we have:

Proposition 3.4. The function $\alpha \to f(\alpha)$ is C^2 and strictly convex on $[\alpha_\infty, \alpha_{-1/2}]$ where α_∞ corresponds to $f'(\alpha_\infty) = \infty$ and $\alpha_{-1/2}$ corresponds to $f'(\alpha_{-1/2}) = -1/2$.

Proposition 3.5. For $\alpha \in [\alpha_\infty, \alpha_{-1/2}]$ we have

$$HD(S_{\alpha+\delta}) = f(\alpha + \delta) + (\alpha + \delta)$$

This is the extension of the theorem proved in ref. 5 for small $|\alpha|$. To $\alpha = 0$ corresponds the almost sure value $\delta \equiv f(\delta) + \delta = 0.9957...$. The dimension of the support being equal to 1 it corresponds to $\bar{\alpha}$ such that $f(\bar{\alpha} + \delta) + (\bar{\alpha} + \delta) = 1$ which corresponds to $\beta = -1$ (not attained here).

Remark. An explicit formula. The ergodic Theorem 2.4, and the explicit iterative formula for $v_\beta(x_0, u)$ given an explicit expression for the local exponent $\alpha + \delta$ as a function of β: $\alpha + \delta = F'(\beta)$, generalizing the formula for the a.s. dimension found in ref. 5. Also, $DH(S_{\alpha+\delta}) = F'(\beta) \times (1 + \beta) - F(\beta)$.

Remark. We can recover from the matrix $M(x)$ the value for $\alpha_{\min}$ and $\alpha_{\max}$, (found in the literature, see ref. 4):

$$\delta + \alpha_{\min} = 0,\, 94042... < \delta + \alpha < \delta + \alpha_{\max} = 1,\, 44042...$$

Indeed, the larger value taken by the limit of sequence $\alpha_n(x)$:

$$\frac{\log \dfrac{|M(x_n) \cdots M(x_0)\, u|}{2^{x_0 + \cdots x_n}}}{\log s^{x_0 + \cdots + x_n}} = \alpha_n(x) + \delta$$

—where s is the golden mean $(-1+\sqrt{5})/2$—corresponds to the case $M(x_n) = \cdots = M(x_0) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ which gives $\delta + \alpha_{\max}(n) = (\log(n+2) - 2n \log 2)/2n \log s$ that is $\delta + \alpha_{\max} = (-\log 2)/\log s = 1.44....$ The smallest values is attained if one choose $M(x_n) = \cdots = M(x_0) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. One has $\delta + \alpha_{\min} = (2 \log((1+\sqrt{5})/2)/4 \log s) - (\log 2/\log s) = 0, 94042....$ The almost sure value $\delta = f(\delta) = 0, 9957...$ corresponds to $\beta = 0$.

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